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MATHEMATICAL MODELING AND OPTIMAL CONTROL OF WHEAT STRIPE RUST *(Yellow Rust)* DISEASE

Siham Hachoum*, Maryam Riouali, Imane Elberrai, Khalid Adnaoui, Noureddine Ouldkhouia

Laboratory Analysis, Modeling and Simulation (LAMS), Department of Mathematics and Computer Science, Faculty of Sciences Ben M'Sik, Hassan II University of Casablanca, Morocco

Abstract. Wheat Stripe Rust is a devastating fungal disease that is the largest limitation to wheat production and threatens the global food supply. In this work, we proposed a mathematical system of ordinary differential equations to model the wheat Stripe Rust transmission dynamics. First, we showed the feasible region, the positivity of the solution, the model's equilibrium points and the basic reproduction number. Second, we used the Pontryagin maximum principle to extend the basic system into an optimal control system by embedding three control measures (resistant cultivars, fungicides and cultural practices). Finally, we used a numerical simulation of the optimality system to demonstrate the thorough impacts of resistant wheat cultivars, fungicide treatments and cultural practices in reducing the epidemic.

Keywords: Stripe Rust disease, Optimal control, Mathematical model.
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Corresponding author: Siham Hachoum, Laboratory Analysis, Modeling and Simulation (LAMS), Department of Mathematics and Computer Science, Faculty of Sciences Ben M'Sik, Hassan II University of Casablanca, Morocco, e-mail: *siham.hachoum-etu@etu.univh2c.ma*
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1 Introduction

Wheat is undoubtedly the world's most important food crop (Morris & Rose, 1996), providing food for billions of people. World wheat production stands at around 781 million metric tons (MMT) during the 2022–2023 year (USDA's). On average, about $20 - 40\%$ of global wheat production is lost due to diseases and pests (Agrios, 2005), which cost the global economy 220 billion dollars each year. Of these diseases, rust is among the most important pathogens, causing a continuous threat to wheat production. Wheat rust diseases (Li & Zeng, 2002) are among the oldest plant diseases known to humans. The rusts are a group of fungal parasite diseases affecting a wide variety of plants, and have the most complicated life cycles of all fungi (Kolmer et al., 2009). There are many types of individual rust diseases (Salgado et al., 2016). One of which is Stripe Rust, is one of the most important wheat diseases that can cause up to 100% yield loss.

Stripe Rust, also known as yellow rust YR (because of its spore color during its infection

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cycle), is a foliar disease of cereals caused by a fungus called Puccinia striiformis (Rodriguez-Algaba et al., 2014) (Pst is a biotrophic fungus that depends on a living host for its development and reproduction (Chen et al., 2014)), is a major infectious disease of soft wheat crops, barley, and durum wheat, which has significantly adverse effects on yield and quality. Yield losses reach up to 70% in cases of severe infection, leading to global losses of over 5 million tons of wheat (Wellings, 2011; Beddow et al., 2015).

In order to work with the Stripe Rust fungus , we must understand how it reproduces and survives (Figure 1). For the Puccinia striiformis (Pst) fungus to replicate, it must go through two phases on two unrelated plants; it alternates between the primary host for an asexual phase in autumn and the alternate host for a sexual phase in spring (Jin et al., 2010). The primary host of these diseases is wheat while the alternate host is typically a weed or native plant. For example, barberry (Berberis vulgaris) serves as the main alternate host for the Stripe Rust fungus. The sexual phase starts at the end of summer, when the spores (Basidiospores) produced

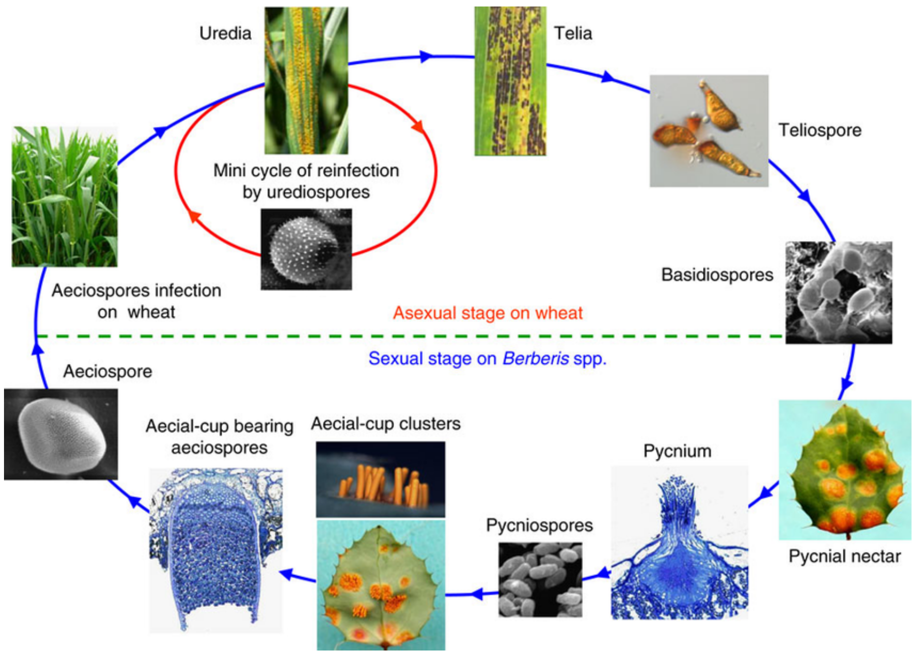

Figure 1: Life cycle of Puccinia striiformis

on wheat infect barberry leaves (Rodriguez-Algaba et al., 2017), Puccinia striiformis then generates pycniospores on the adaxial surface of leaves, ensuring its reproduction and survival even at very low temperatures (down to -10^0C). In spring, as the weather becomes cool and damp, the disease is dispersed through wind-blown spores, and the fungus resumes its development to initiate the asexual phase. Spores (aeciospores) produced on barberry plants infect wheat leaves, initiating secondary contamination through the production of new uredospores. The fungus penetrates the cell wall and enters the plant's tissue to extract nutrients, causing cell death. This process (phase) takes from 6 to 10 days or longer depending on temperature, until the symptoms of Stripe Rust appear on the leaf surface.When the disease develops on wheat, the spores (uredospores) produced can cause auto-infection where spores infect the same plants on which they were produced through simple contact or be carried by the wind. This spore stage of the life cycle is known as the repeating stage and is responsible for the rapid development of disease outbreaks.

The management of Stripe Rust fungus can be achieved (El Khoury & Makkouk, 2010; Van Der Plank, 1963) through three main control options (Chen, 2005): cultivating resistant wheat cultivars, applying fungicide seed treatments and foliar fungicide treatments before the disease has developed significantly, and adjusting certain cultural practices, such as timely planting, sanitation, and crop rotation. By implementing these measures, farmers can minimize the impact of Puccinia striiformis and ensure the sustainable production of cereal crops (Chen & Kang, 2017).

Over the years, mathematical models have proved to be reliable and powerful tools used to understand the dynamics of disease transmission and in decision-making processes regarding intervention programs for implementing control measures to prevent and minimize the impact of infectious diseases. Mathematical models can also be employed to analyze the dynamics of plant diseases (Bazarra et al., 2022). Many mathematical models have been constructed and analyzed to describe the transmission dynamics of various plant diseases.

The structure of this paper is arranged as follows: In Section 2, we propose a formulation of the Stripe Rust model, stating the definitions of the various parameters of the model. In Section 3, the model properties and the analysis of the equilibrium points are illustrated. In Section 4, we state the control problem of the Stripe Rust transmission model, then we apply Pontryagin's Maximum Principle to find the necessary conditions for the optimal control. In Section 5, we conduct the numerical simulation in Matlab to prove our theoretical results, and a brief discussion is also provided in this section. Finally, in Section 6, we give conclusions.

2 The Model Formulation and Description

2.1 Formulation

In this section, we developed a Stripe Rust transmission dynamics model in which the total main plant population is represented by the $SEIRS$ model, while the population of the alternate host-plants is described by the $S_hE_hI_h$ model.

↠ The total population of main plants at time t, denoted by $P(t)$ is classified into the susceptible plants class (S), exposed/latent plants class (E), infectious plants class (I), and recovered plants class (R).
Thus, the total population of main plants is given by

$$P(t) = S(t) + E(t) + I(t) + R(t).$$

↠ The total alternative host-plants population at time t, denoted by $H(t)$, is subdivided into the susceptible alternative host-plants (S_h), exposed/latency alternative host-plants (E_h), and the infectious alternative host-plants (I_h).
Hence, the total population of alternative host-plants is given by

$$H(t) = S_h(t) + E_h(t) + I_h(t).$$

The population flow among these compartments is presented in Figure 2.

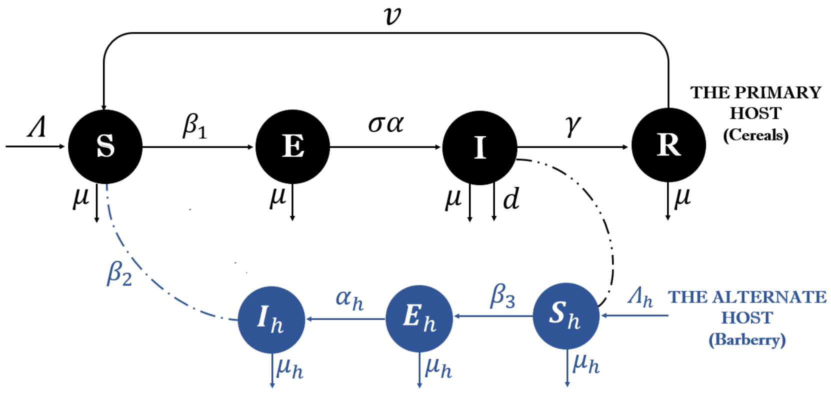

Figure 2: Diagram of Stripe Rust transmission dynamics

: The blue arrows represent a alternative host-plant moving from one class to the next.

: The black arrows represent a main host-plant moving from one class to the next.

: The blue dashed lines indicate the direction of infection from an infectious alternative host-plants (I_h) to a susceptible main host-plants (S).

: The black dashed lines indicate the direction of infection from an infectious main host-plants (I) to a susceptible alternative host-plants (S_h).

Based on the assumptions and the diagram in Figure 2, the deterministic model that describes the transmission dynamics of Stripe Rust disease is represented by the following nonlinear differential equations

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \mu S - (\beta_1 I + \beta_2 I_h)S + \nu R, & S(0) = S_0 \geqslant 0 \\ \dfrac{dE}{dt} = (\beta_1 I + \beta_2 I_h)S - (\sigma\alpha + \mu)E, & E(0) = E_0 \geqslant 0 \\ \dfrac{dI}{dt} = \sigma\alpha E - (\gamma + d + \mu)I, & I(0) = I_0 \geqslant 0 \\ \dfrac{dR}{dt} = \gamma I - (\nu + \mu)R, & R(0) = R_0 \geqslant 0 \\ \dfrac{dS_h}{dt} = \Lambda_h - \beta_3 I S_h - \mu_h S_h, & S_h(0) = S_{h(0)} \geqslant 0 \\ \dfrac{dE_h}{dt} = \beta_3 I S_h - (\alpha_h + \mu_h)E_h, & E_h(0) = E_{h(0)} \geqslant 0 \\ \dfrac{dI_h}{dt} = \alpha_h E_h - \mu_h I_h, & I_h(0) = I_{h(0)} \geqslant 0. \end{cases} \tag{1}$$

We assume that all of the parameters in the model are positive.

2.2 Model Interpretation

The values of the different parameters used in the model (1) are shown in the Table 1.
↠ We have deemed that the parameters listed are positive.

Table 1: The definitions of parameters in model (1)

Parameter	***Description***
$\boldsymbol{\Lambda}$	The birth rate.
$\boldsymbol{\Lambda_h}$	Replenishing rate of alternative host plants.
$\boldsymbol{1/\sigma}$	Incubation period.
$\boldsymbol{\beta_1}$	The rate at which the infected main host plants transmits the Pst infections to the susceptible main host plants.
$\boldsymbol{\beta_2}$	The rate at which the infected alternative host plants transmits the Pst infections to the susceptible main host plants.
$\boldsymbol{\beta_3}$	The rate at which the infected main host plants transmits the Pst infections to the susceptible alternative host plants.
$\boldsymbol{\alpha}$	The rate at which the exposed main host plants complete their incubation period and become infected.
$\boldsymbol{\alpha_h}$	The rate at which the alternative host plants that have been exposed become infected.
$\boldsymbol{\gamma}$	Recovery rate of Strip Rust from infectious main plants.
$\boldsymbol{\mu}$	The natural deaths rate in all classes of main plants.
$\boldsymbol{\mu_h}$	Death rate of alternative host plants.
$\boldsymbol{d}$	Death rate of infected main plants due to the disease.
$\boldsymbol{\nu}$	Rate at which recovered main plants returns back to susceptible class (grassy weeds).

3 Mathematical Analysis of the Model

3.1 Positivity of the Solution

The Stripe Rust model (1) has non-negative initial data $(S(0), E(0), I(0), R(0), S_h(0), E_h(0), I_h(0)) \geqslant 0$ for $t \geqslant 0$.

Consider the first equation from the system (1)

$$\frac{dS}{dt} = \Lambda - \mu S - (\beta_1 I + \beta_2 I_h)S + \nu R. \tag{2}$$

This leads to

$$\frac{dS}{dt} \geqslant -(\mu + (\beta_1 I + \beta_2 I_h))S. \tag{3}$$

Integrating equation (3), we obtain

$$S(t) \geqslant S(0)e^{-(\mu+\beta_1 I+\beta_2 I_h)t} > 0. \tag{4}$$

Thus, $S(t)$ will be non-negative for $t \geqslant 0$.

Similarly, we can prove the other state variables $(E(t), I(t), R(t), S_h(t), E_h(t), I_h(t)) > 0$ for all $t \geqslant 0$.

Consequently, the Stripe Rust transmission model is both epidemiologically meaningful.

3.2 Invariant Region

The model (1) involves two types of plant populations, the main plants population $P(t)$ and the alternative host-plants population $H(t)$.

The total populations of them from system (1) are given respectively by

$$\begin{cases} P(t) = S(t) + E(t) + I(t) + R(t), \\ H(t) = S_h(t) + E_h(t) + I_h(t), \end{cases} \tag{5}$$

Differentiation of (5), with respect to time gives

$$\begin{cases} \dfrac{dP}{dt} = \dfrac{dS(t)}{dt} + \dfrac{dE(t)}{dt} + \dfrac{dI(t)}{dt} + \dfrac{dR(t)}{dt}, \\ \dfrac{dH}{dt} = \dfrac{dS_h(t)}{dt} + \dfrac{dE_h(t)}{dt} + \dfrac{dI_h(t)}{dt}, \end{cases} \tag{6}$$

Adding the first four equations and the last three equations of the model gives

$$\begin{cases} \dfrac{dP}{dt} = \Lambda - \mu P - dI \quad \leqslant \Lambda - \mu P, \\ \dfrac{dH}{dt} = \Lambda_h - \mu_h H, \end{cases} \tag{7}$$

By applying Birkhoff and Rota's (1982) theory of differential inequality, obtain the following result

$$0 \leqslant P(t) \leqslant P(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t}) \quad \text{and} \quad 0 \leqslant H(t) \leqslant H(0)e^{-\mu_h t} + \frac{\Lambda_h}{\mu_h}(1 - e^{-\mu_h t}).$$

And therefore,

$$P(t) \leqslant \frac{\Lambda}{\mu} \quad \text{and} \quad H(t) \leqslant \frac{\Lambda_h}{\mu_h} \quad \text{as} \quad t \to +\infty.$$

The invariant region of system (1) is given by

$$\Omega = \Omega_1 + \Omega_2,$$

where Ω_1 is the invariant region for main plants population, and Ω_2 is the invariant region for the alternative host-plants population.

$$\begin{cases} \Omega_1 = \{(S, E, I, R) \in \mathbb{R}^4_+ : P(t) \leqslant \dfrac{\Lambda}{\mu}\}, \\ \Omega_2 = \{(S_h, E_h, I_h) \in \mathbb{R}^3_+ : H(t) \leqslant \dfrac{\Lambda_h}{\mu_h}\}, \end{cases}$$

Thus, the system's invariant region of model (1) is given by

$$\Omega = \{(S, E, I, R, S_h, E_h, I_h) \in \mathbb{R}^7_+ : P(t) \leqslant \frac{\Lambda}{\mu},\ H(t) \leqslant \frac{\Lambda_h}{\mu_h}\}.$$

We can deduce that all feasible solutions of the system (1) are bounded in a positive invariant region Ω (Hethcote, 2000).

3.3 Disease Free Equilibrium (DFE)

By setting all equations of model (1) equal to zero with $E = 0$, $I = 0$ and $E_h = 0$, $I_h = 0$, we obtain the single Stripe Rust-free equilibrium point defined by

$$E_0 = (S^0, E^0, I^0, R^0, S^0_h, E^0_h, I^0_h) = (\frac{\Lambda}{\mu}, 0, 0, 0, \frac{\Lambda_h}{\mu_h}, 0, 0).$$

3.4 Basic Reproduction Number

The basic reproduction number R_0 (Diekmann et al., 1990; Hyman & Li, 2000) is defined as the average number of secondary infection caused by a primary infection during a given time period (Van den Driessche & Watmough, 2002).

To determine R_0, we need to define the next generation matrix (NGM). A next-generation matrix K_L consists of two parts: the matrix U corresponds to transmissions and the inverse of matrix V corresponds to transitions. We consider four compartments E, I, E_h and I_h which contribute to new infections or the secondary cases of main plants and alternative host-plants;

$$\begin{cases} \dfrac{dE}{dt} = (\beta_1 I + \beta_2 I_h)S - (\sigma\alpha + \mu)E, \\ \dfrac{dI}{dt} = \alpha E - (\gamma + d + \mu)I, \\ \dfrac{dE_h}{dt} = \beta_3 I S_h - (\alpha_h + \mu_h)E_h, \\ \dfrac{dI_h}{dt} = \alpha_h E_h - \mu_h I_h. \end{cases} \tag{8}$$

Let $Z = (E, I, E_h, I_h)^T$, our system can be written as $Z' = u - v$.

The new infection matrix u and the transition matrix v are defined by the column matrices

$$u = \begin{pmatrix} (\beta_1 I + \beta_2 I_h)S \\ 0 \\ \beta_3 I S_h \\ 0 \end{pmatrix} \quad \text{and} \quad v = -\begin{pmatrix} (\sigma\alpha + \mu)E \\ (\gamma + d + \mu)I - \alpha E \\ (\alpha_h + \mu_h)E_h \\ \mu_h I_h - \alpha_h E_h \end{pmatrix}$$

The Jacobian of these matrices at Strip Rust-free equilibrium points are given by

$$U = \begin{pmatrix} 0 & \beta_1\Lambda/\mu & 0 & \beta_2\Lambda/\mu \\ 0 & 0 & 0 & 0 \\ 0 & \beta_3\Lambda_h/\mu_h & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -(\sigma\alpha + \mu) & 0 & 0 & 0 \\ \sigma\alpha & -(\gamma + d + \mu) & 0 & 0 \\ 0 & 0 & -(\alpha_h + \mu_h) & 0 \\ 0 & 0 & \alpha_h & -\mu_h \end{pmatrix}$$

Where

$$V^{-1} = \begin{pmatrix} \dfrac{-1}{(\sigma\alpha + \mu)} & 0 & 0 & 0 \\ \dfrac{-\sigma\alpha}{(\gamma + d + \mu)(\mu + \sigma\alpha)} & \dfrac{-1}{(\gamma + d + \mu)} & 0 & 0 \\ 0 & 0 & \dfrac{-1}{(\alpha_h + \mu_h)} & 0 \\ 0 & 0 & \dfrac{-\alpha_h}{\mu_h(\alpha_h + \mu_h)} & \dfrac{-1}{\mu_h} \end{pmatrix}$$

Multiplying U with V^{-1}

$$UV^{-1} = \begin{pmatrix} \dfrac{-\beta_1\sigma\alpha\Lambda}{\mu(\sigma\alpha + \mu)(\gamma + d + \mu)} & \dfrac{-\beta_2\Lambda}{\mu(\gamma + d + \mu)} & \dfrac{-\beta_1\alpha_h\Lambda}{\mu\mu_h(\alpha_h + \mu_h)} & \dfrac{-\beta_2\Lambda}{\mu\mu_h} \\ 0 & 0 & 0 & 0 \\ \dfrac{-\beta_3\sigma\alpha\Lambda_h}{\mu_h(\sigma\alpha + \mu)(\gamma + d + \mu)} & \dfrac{-\beta_3\Lambda_h}{\mu_h(\gamma + d + \mu)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the NGM (Diekmann et al., 2010) with large domain K_L is four-dimensional and given by

$$K_L = -UV^{-1} = \begin{pmatrix} \frac{\beta_1\sigma\alpha\Lambda}{\mu(\sigma\alpha+\mu)(\gamma+d+\mu)} & \frac{\beta_2\Lambda}{\mu(\gamma+d+\mu)} & \frac{\beta_1\alpha_h\Lambda}{\mu\mu_h(\alpha_h+\mu_h)} & \frac{\beta_2\Lambda}{\mu\mu_h} \\ 0 & 0 & 0 & 0 \\ \frac{\beta_3\sigma\alpha\Lambda_h}{\mu_h(\sigma\alpha+\mu)(\gamma+d+\mu)} & \frac{\beta_3\Lambda_h}{\mu_h(\gamma+d+\mu)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The dominant eigenvalue corresponding to the Spectral radius of this matrix K_L is equal to R_0 the basic reproduction number, where $\rho(K_L) = R_0$

$$R_0 = \frac{\beta_1\sigma\alpha\Lambda}{2\mu(\sigma\alpha+\mu)(\gamma+d+\mu)}\left(1+\sqrt{1+\frac{4\beta_3\Lambda_h\alpha_h\mu(\sigma\alpha+\mu)(\gamma+d+\mu)}{\beta_1\mu_h^2\Lambda\alpha\sigma(\alpha_h+\mu_h)}}\right) \tag{9}$$

Furthermore, model (1) has the endemic equilibrium $E_1 = (S^*, E^*, I^*, R^*, S_h^*, E_h^*, I_h^*)$.

3.5 Endemic Equilibrium Point

The Endemic Equilibrium $E_1 = (S^*, E^*, I^*, R^*, S_h^*, E_h^*, I_h^*)$ of system (1) is determined by

$$\begin{aligned} S^* &= \frac{1}{\mu}\left(\Lambda + \nu\frac{\gamma(\Lambda-\mu P)}{d(\mu+\nu)} - (\sigma\alpha+\mu)\frac{(\gamma+\mu+d)(\Lambda-\mu P)}{\sigma\alpha d}\right), \\ E^* &= \frac{(\gamma+\mu+d)(\Lambda-\mu P)}{\sigma\alpha d}, \\ I^* &= \frac{(\Lambda-\mu P)}{d}, \\ R^* &= \frac{\gamma(\Lambda-\mu P)}{d(\mu+\nu)}, \\ S_h^* &= \frac{d\Lambda_h}{(\beta_3(\Lambda-\mu P)+\mu_h d)}, \\ E_h^* &= \frac{\Lambda_h}{(\alpha_h+\mu_h)(1-\frac{\mu_h d}{\beta_3(\Lambda-\mu P)})}, \\ I_h^* &= \frac{\alpha_h\Lambda_h}{\mu_h(\alpha_h+\mu_h)(1-\frac{\mu_h d}{\beta_3(\Lambda-\mu P)})}. \end{aligned}$$

The endemic equilibrium E_1 exists if $R_0 > 1$.

4 Optimal Control Model

In this section, we will analyze and extend the Stripe Rust transmission model (1) using optimal control strategies.

4.1 Problem Formulation

Our goal is to minimize the impact of Puccinia striiformis by reducing the number of exposed and infected plants in the crops. To achieve this, we extended the Stripe Rust model (1) to an optimal control problem to determine strategic control decisions using a mathematical model of biological situations (Lenhart & Workman, 2007).

By incorporating the following control measures into the model (1);

- $\boldsymbol{c_1}$ - Adjusting cultural practices.
- $\boldsymbol{c_2}$ - Represent the control on the use a specific type of fungicide that controls disease.
- $\boldsymbol{c_3}$ - Growing resistant cultivars (choosing a variety resistant to Stripe Rust).

The optimal control model is formulated as the following nonlinear systems

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - (1-c_1)(\beta_1 I + \beta_2 I_h)S - (c_2+\mu)S + \nu R, \\ \dfrac{dE}{dt} = (1-c_1)(\beta_1 I + \beta_2 I_h)S - (\mu + \sigma\alpha)E, \\ \dfrac{dI}{dt} = \sigma\alpha E - (\gamma + d + \mu + c_3)I, \\ \dfrac{dR}{dt} = (\gamma + c_3)I - (\mu+\nu)R, \\ \dfrac{dS_h}{dt} = \Lambda_h - \beta_3 I S_h - \mu_h S_h, \\ \dfrac{dE_h}{dt} = \beta_3 I S_h - (\alpha_h + \mu_h)E_h, \\ \dfrac{dI_h}{dt} = \alpha_h E_h - \mu_h I_h, \end{cases} \tag{10}$$

The effect of the optimal control model is determined by the following objective function (Lenhart & Workman, 2007)

$$J(c_1, c_2, c_3) = \int_0^{t_{end}} \left(A_1 E + A_2 I + \frac{1}{2}\big(B_1 c_1^2 + B_2 c_2^2 + B_3 c_3^2\big) \right) dt, \tag{11}$$

Where

- t_{end} denoted the terminal time.
- A_1 and A_2 are the positive constants coefficients for the exposed and infected plants respectively.
- The constants B_1, B_2 and B_3 are the weight constants coefficients for the control variables c_1, c_2 and c_3 respectively.

Our main goal of this problem is to find an optimal control functions $c_i^*(t)$ for $i = (1, \dots, 3)$.

$$J(c_1^*, c_2^*, c_3^*) = \min\big\{J(c_1, c_2, c_3),\ (c_1, c_2, c_3) \in C\big\}.$$

Where C is set of admissible control to the model (10) (The control set).

$$C = \big\{(c_1(.), c_2(.), c_3(.));\ 0 \leqslant c_1(t), c_2(t), c_3(t) \leqslant 1, \forall t \in [0, t_{end}]\big\}.$$

4.2 Existence of Optimal Control Problem

In this part, we prove the existence of an optimal control triples that optimize the objective functional (11) by applying the theorem.

Theorem 1. *There exists an optimal control* $c^* = (c_1^*, c_2^*, c_3^*) \in C$ *such that; the control model (10) with initial conditions at* $t = 0$ *and* $J(c_1^*, c_2^*, c_3^*) = \min\big\{J(c_1, c_2, c_3),\ (c_1, c_2, c_3) \in C\big\}$.

Proof. The existence of an optimal control pair is proved under the following conditions given in (Fleming & Rishel, 1975):

1. The control set C is convex, bounded and closed in $L^2(0, t_{end})$.
   ↠ *The set C is bounded closed and convex (Beck, 2014) by definition, so the condition is verified.*

2. The control set and control variables is non-empty.
   ↠ *These conditions are verified by a result of (Lukes, 2003) which ensures the existence of solutions for the state system (10) with constant coefficients.*

3. The right side of the state system is bounded by a linear function of control variables depending on time and state variables.
   ↠ *The state system (10) is clearly linear in control variables c_1, c_2 and c_3 with coefficients depending on state variables.*

4. The integrand of the objective functional (11) is convex on the C and there exist positive numbers Q_1, Q_2 and a constant $\varepsilon > 1$ such that the integrand of the objective function is bounded by $Q_1\big(|c_1|^2 + |c_2|^2 + |c_3|^2\big)^{\varepsilon/2} - Q_2$.
   ↠ *The integrand of the objective functional* $A_1E + A_2I + \frac{1}{2}\sum_{i=1}^{3} B_i c_i^2$ *is the sum of convex functions* (Chong & Zak, 2004) *and hence convex with respect to control variables.*
   ↠ *Lastly, we have*

$$
\begin{aligned}
A_1E + A_2I + \frac{1}{2}\sum_{i=1}^{3} B_i c_i^2 &\geqslant \frac{1}{2}\big(B_1c_1^2 + B_2c_2^2 + B_3c_3^2\big) \\
&\geqslant \frac{1}{2}\left(B_1c_1^2 + B_2c_2^2 + B_3c_3^2\right)^{\varepsilon/2} - Q_2, \\
&\geqslant Q_1\left(|c_1|^2 + |c_2|^2 + |c_3|^2\right)^{\varepsilon/2} - Q_2.
\end{aligned}
$$

Therefore, the state variables are bounded and the existence of optimal control of the model (10) is concluded.

□

4.3 Characterization of the Optimal Control

In this part, we present optimality conditions for the optimal control problem (10) and detail its properties. On the basis of Pontryagin's Maximum Principle (PMP) (Pontryagin et al., 1962), We define the Hamiltonian function of the optimal control problem (10) as follows

$$
H = L(E, I, c_i) + \lambda_j \left(\frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR}{dt} + \frac{dS_h}{dt} + \frac{dE_h}{dt} + \frac{dI_h}{dt}\right). \tag{12}
$$

With the Lagrangian:

$$
L(E, I, c_i) = A_1E + A_2I + \frac{1}{2}\sum_{i=1}^{3} B_i c_i^2. \tag{13}
$$

Substituting the Lagrangian (13) and the optimal control model (10) into the Hamiltonian (12), we obtain

$$\begin{aligned}
H &= A_1E + A_2I + \frac{1}{2}\big(B_1c_1^2 + B_2c_2^2 + B_3c_3^2\big) \\
&+ \lambda_1\Big[\Lambda - (1-c_1)\beta_1SI - (1-c_1)\beta_2SI_h - (c_2+\mu)S + \nu R\Big] \\
&+ \lambda_2\Big[(1-c_1)\beta_1SI + (1-c_1)\beta_2SI_h - (\mu+\sigma\alpha)E\Big] \\
&+ \lambda_3\Big[\sigma\alpha E - (\gamma+d+\mu+c_3)I\Big] \\
&+ \lambda_4\Big[(\gamma+c_3)I - (\mu+\nu)R\Big] \\
&+ \lambda_5\Big[\Lambda_h - (\beta_3I+\mu_h)S_h\Big] \\
&+ \lambda_6\Big[\beta_3IS_h - (\alpha_h+\mu_h)E_h\Big] \\
&+ \lambda_7\Big[\alpha_hE_h - \mu_hI_h\Big].
\end{aligned} \tag{14}$$

Where the symbols λ_j for $j = (1,\dots,7)$ represent the adjoint variables associated to the state variables.

For optimized solution (x, c) of optimal control problem (10), $\exists$ a non-trivial vector function $\lambda(t) = (\lambda_1(t),\dots,\lambda_7(t))$ satisfying the following equations

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\partial}{\partial\lambda}H(t,x,c,\lambda), \\
0 &= \frac{\partial H(t,x,c,\lambda)}{\partial c}. \\
\lambda' &= -\frac{\partial}{\partial x}H(t,x,c,\lambda)
\end{aligned} \tag{15}$$

Where $x = (S, E, I, R, S_h, E_h, I_h)$ and $c = (c_1, c_2, c_3)$.
Therefore, we can now apply the necessary conditions to the Hamiltonian H.

The adjoint system is determined by applying the first and third equations in (15) into (14), with respect to each state variables

$$\begin{cases}
\dfrac{d\lambda_1(t)}{dt} = -\dfrac{\partial H}{\partial S} = (\lambda_1-\lambda_2)(1-c_1)(\beta_1I+\beta_2I_h) + \lambda_1(c_2+\mu), \\
\dfrac{d\lambda_2(t)}{dt} = -\dfrac{\partial H}{\partial E} = -A_1 + \lambda_2(\mu+\sigma\alpha) - \lambda_3\sigma\alpha, \\
\dfrac{d\lambda_3(t)}{dt} = -\dfrac{\partial H}{\partial I} = -A_2 + (\lambda_1-\lambda_2)(1-c_1)\beta_1S + (\lambda_3-\lambda_4)(\gamma+c_3) + \lambda_3(d+\mu) + (\lambda_5-\lambda_6)\beta_3S_h, \\
\dfrac{d\lambda_4(t)}{dt} = -\dfrac{\partial H}{\partial R} = \nu(\lambda_4-\lambda_1) + \lambda_4\mu, \\
\dfrac{d\lambda_5(t)}{dt} = -\dfrac{\partial H}{\partial S_h} = (\lambda_5-\lambda_6)\beta_3I + \lambda_5\mu_h, \\
\dfrac{d\lambda_6(t)}{dt} = -\dfrac{\partial H}{\partial E_h} = \lambda_6(\alpha_h+\mu_h) - \lambda_7\alpha_h, \\
\dfrac{d\lambda_7(t)}{dt} = -\dfrac{\partial H}{\partial I_h} = (\lambda_1-\lambda_2)(1-c_1)\beta_2S + \lambda_7\mu_h.
\end{cases} \tag{16}$$

With transversality conditions,

$$\lambda_j(t_{end}) = 0, \quad \text{for} \qquad j = (1,\dots,7).$$

To obtain the optimal control value, we apply the second equation in (15) by the partial derivative of Hamiltonian (14) with respect to control variable

$$\frac{\partial H}{\partial c_i} = 0 \ , \quad \text{for} \qquad i = 1, 2, 3.$$

Which implies that

$$\begin{aligned} \frac{\partial H}{\partial c_1} &= B_1 c_1 + S(\lambda_1 - \lambda_2)(\beta_1 I + \beta_2 I_h) &&= 0, \\ \frac{\partial H}{\partial c_2} &= B_2 c_2 - \lambda_1 S &&= 0, \\ \frac{\partial H}{\partial c_3} &= B_3 c_3 + I(\lambda_4 - \lambda_3) &&= 0. \end{aligned} \tag{17}$$

From (17), we have the controls

$$\begin{aligned} c_1^* &= \frac{S(\lambda_2 - \lambda_1)(\beta_1 I + \beta_2 I_h)}{B_1}, \\ c_2^* &= \frac{\lambda_1 S}{B_2}, \\ c_3^* &= \frac{I(\lambda_3 - \lambda_4)}{B_3}. \end{aligned} \tag{18}$$

Finally, the optimal controls $c^* = (c_1^*, c_2^*, c_3^*)$ with the boundary condition can be written as

$$\begin{aligned} c_1^* &= max\left(\{0, min\{1, \frac{S(\lambda_2 - \lambda_1)(\beta_1 I + \beta_2 I_h)}{B_1}\}\}\right), \\ c_2^* &= max\left(\{0, min\{1, \frac{\lambda_1 S}{B_2}\}\}\right), \\ c_3^* &= max\left(\{0, min\{1, \frac{I(\lambda_3 - \lambda_4)}{B_3}\}\}\right). \end{aligned}$$

Next, we will see the simulation of the optimality system (10).

5 Numerical Simulation

Wheat Stripe Rust poses a significant threat to global wheat production, jeopardizing the stability of the world's food supply. This study aims to provide a precise understanding of disease dynamics and, more importantly, to optimize control strategies for mitigating the impact of wheat Stripe Rust, using mathematical modeling and optimal control systems.

In this section, we present numerical simulations (Shampine et al., 2003) describing the dynamics of the respective plant compartments with and without control. This demonstration aims to showcase the effectiveness of our strategy in mitigating the impact of Stripe Rust on cereal crops. Through numerical simulations, we highlight the tangible effects of control strategies represented in three pivotal measures: resistant cultivars, fungicides, and cultural practices.

Additionally, we provide valuable insights for policymakers, researchers, and practitioners aiming to enhance wheat crop resilience and ensure global food security in the face of the widespread threat posed by wheat Stripe Rust.

5.1 Graphical Results

⁎ **Without controls:** In order to demonstrate the efficiency of model (1), we present graphical results without controls, allowing us to observe how the growth results align with reality to some extent (see Figures 3 and 4).

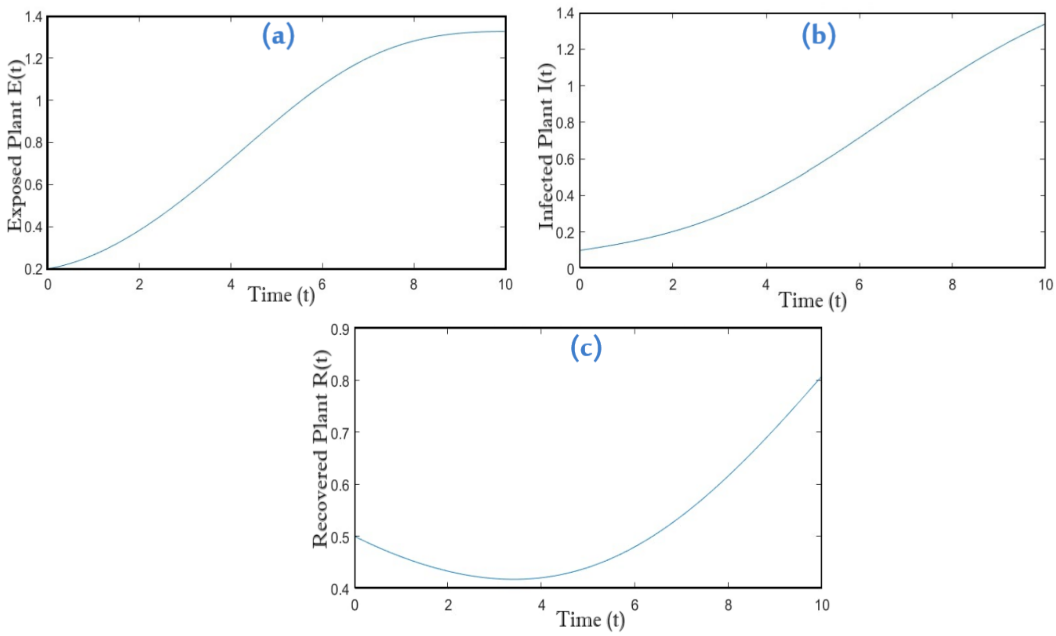

Figure 3: The graphical results of the main plants without controls

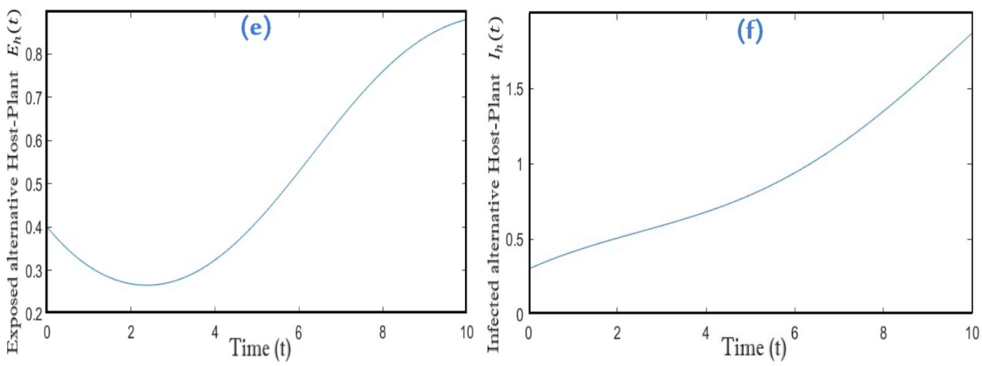

Figure 4: The graphical results of the alternative host-plants without controls

⁎ **With controls:** Using the system (10), we implement all the control variables proposed at the same time, we analyzed the impact of growing resistant cultivars c_1, fungicide treatments c_2, and adjusting some cultural practices c_3. The resulting graphical representations are displayed in the following Figures.

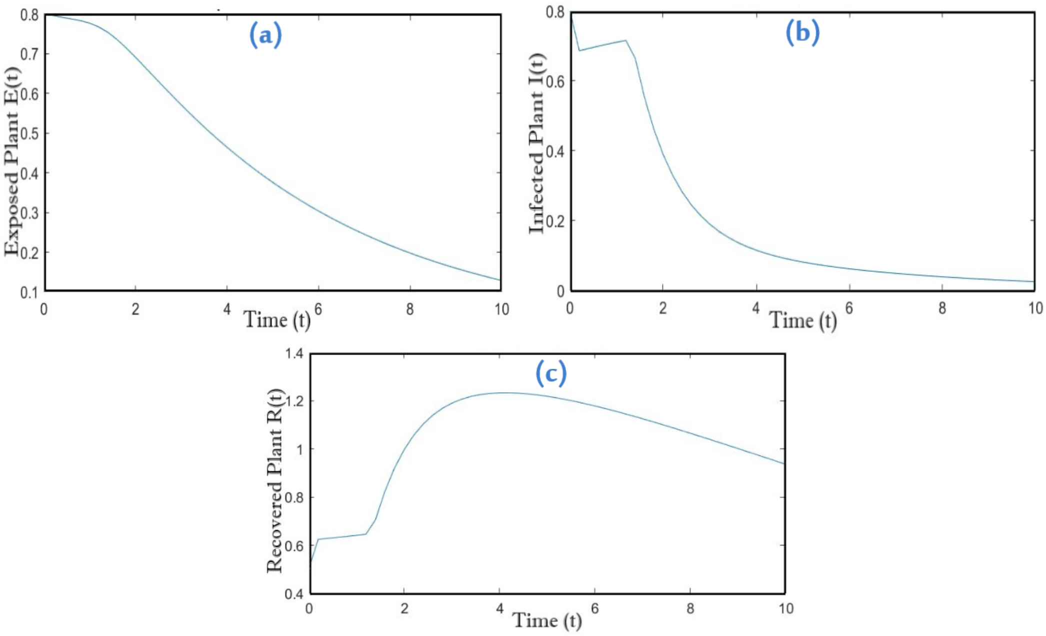

Figure 5: The graphical results of the main plants with controls

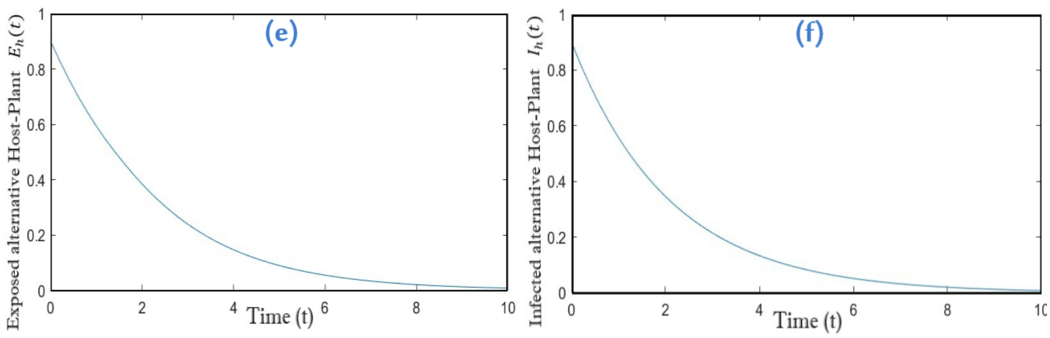

Figure 6: The graphical results of the alternative host-plants with controls

5.2 Discussion of the results

• **Looking at Figure 3 and 5.**

Analyzing the graphical representations provides intriguing insights into the effectiveness of control measures. From figures 3(a) and 3(b), it is clear that the number of exposed and infected plants exhibits an increasing trajectory in the absence of any control interventions. This confirms that Stripe Rust fungus infection poses a major threat to crops, as it can cause significant losses in productivity and financial returns for farmers, leading us to the need to improve and intensify control efforts.

A remarkable result is shown in figures 5(a) and 5(b), where the addition of control measures results in a notable and significant decrease in the number of exposed and infected plants. This striking difference highlights the effectiveness of the implemented control measures and their direct impact in significantly mitigating the spread of infection among plant populations. It is also important to note that while the number of plants recovered slowly increases in the absence of control variables, this number increases considerably in the optimal system (Figure 5(c)). This reinforces the importance of rapid intervention in managing plant health and protecting agricultural productivity.

• **Looking at Figure 4 and 6.**
The optimal control strategy for alternative host plants is shown in figures 6(e) and 6(f). Interestingly, although control strategies are not directly applied to exposed and infected alternative host plants, their numbers show a marked decrease compared to the case where these strategies were not implemented (Figure 4). This decrease can most likely be attributed to changes in certain specific cultural practices (c_3). Indeed, additional analyses reveal that these modifications have a significant impact on the dynamics of alternative host plants, which highlights the importance of taking into account specific cultural practices in the design of control strategies. This underscores the need for a comprehensive understanding of the intricate relationships between control measures and specific plant species in agricultural systems.

6 Conclusion

In this paper, we develop a mathematical model of Stripe Rust disease using a deterministic system of differential equations. Initially, we explore the feasible region and positivity of the solution of the model. Subsequently, we determine the model's equilibrium points and the basic reproduction number using the next-generation matrix method.

Additionally, we extend the Stripe Rust transmission model to an optimal control problem by incorporating three control strategies (resistant cultivars, fungicides, and cultural practices). We utilize the Pontryagin maximum principle to derive the expression for optimal control.

Based on numerical analysis, we propose that the combination of cultivating resistant wheat cultivars, implementing fungicide treatments prior to the onset of the disease, and adjusting certain cultural practices is the most effective strategy for reducing the number of infected wheat plants and effectively mitigating Stripe Rust.

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